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Asymptotic behaviour of solutions of real two-dimensional differential system with nonconstant delay in an unstable case

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Abstract

The asymptotic behaviour for the solutions of a real two-dimensional system with a bounded nonconstant delay is studied under the assumption of instability. Our results improve and complement previous results by J. Kalas, where the sufficient conditions assuring the existence of bounded solutions or solutions tending to origin for t approaching infinity are given. The method of investigation is based on the transformation of the considered real system to one equation with complex-valued coefficients. Asymptotic properties of this equation are studied by means of a suitable Lyapunov-Krasovskii functional and by virtue of the Ważewski topological principle.

Mathematics Subject Classification: 34K12, 34K20

Key Words and Phrases: Delayed differential equations, Asymptotic behaviour, Boundedness of solutions, Lyapunov method, Ważewski topological principle.

1 Introduction

Consider the real two-dimensional system

$$x'(t) = \mathsf{A}(t)x(t) + \mathsf{B}(t)x(\theta(t)) + h(t, x(t), x(\theta(t))), \tag{0}$$

where $\theta(t)$ is a real-valued function, $\mathsf{A}(t) = (a_{jk}(t))$, $\mathsf{B}(t) = (b_{jk}(t))$ $(j, k = 1, 2)$ are real square matrices and $h(t, x, y) = (h_1(t, x, y), h_2(t, x, y))$ is a real vector-valued function, $x = (x_1, x_2)$, $y = (y_1, y_2)$. The functions θ, a_{jk} are supposed to be locally absolutely continuous on $[t_0, \infty)$, b_{jk} are locally Lebesgue integrable on $[t_0, \infty)$ and the function h satisfies Carathéodory conditions on $[t_0, \infty) \times \mathbb{R}^4$. Moreover, the uniqueness property for solutions of (0) is supposed through the paper.

There is a lot of papers dealing with the stability and asymptotic behaviour of n-dimensional real vector equations with delay. Since the plane has special topological properties different from those of n-dimensional space, where $n \geq 3$

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or $n = 1$, it is interesting to study asymptotic behaviour of two-dimensional systems by using tools which are typical and effective for two-dimensional systems. The convenient tool is the combination of the method of complexification and the method of Lyapunov-Krasovskii functional. The method of complexification is based on the transformation of (0) to an equation with complex conjugate coordinates. This method, together with the use of a convenient Lyapunov-Krasovskii functional and a Razumikhin-type version of Ważewski topological principle, enables to simplify some considerations and estimations and it leads to new, effective and easy applicable results on stability, asymptotic stability, instability or boundedness of solutions of the system (0).

Remark that the Razumikhin-type version of Ważewski topological principle for retarded functional differential equations was formulated in papers of K. P. Rybakowski [20], [21] and there is a number of papers using Ważewski topological method for the investigation of the asymptotic properties of solutions of both ordinary and delayed differential equations; we mention here only some papers by J. Diblík and his collaborators [2], [3], [5], [6], [7]. Finally, notice that complex differential systems were used also by further authors for the solution of various problems related to differential equations, see e. g. the papers of J. Mawhin [17], J. Campos and J. Mawhin [1] and of R. Manàsevich, J. Mawhin, F. Zanolin [14], [15], [16].

However, it seems that there are no results concerning the existence of bounded solutions in a "small" neighbourhood of the origin for the system (0) under the condition of instability. This paper, as a continuation of our previous papers, brings new results of this type.

Stability and asymptotic properties of the solutions for the stable case of (0) are investigated in [11], [19]. The asymptotic properties for the solutions of the equation with a constant delay under the condition of instability were studied in [8], [9]. The similar results for an ordinary differential equation can be found in [13]. In [10], the results of [8] were generalized to the equation (0) with a bounded nonconstant delay. In [9], it was shown that it is useful to investigate (0) also under different conditions, namely the conditions, when the shortened equation $x'(t) = \mathsf{A}(t)x(t)$ is closer to a "focus" than to a "node" at origin. In the present paper, which is related to paper [10], we examine (0) under these assumptions.

The motivation is to improve the results presented in [10], to generalize the results of [9] and to illustrate the advancement and applicability with several examples.

Introducing complex variables $z = x_1 + ix_2$, $w = y_1 + iy_2$, we can rewrite the system (0) into an equivalent equation with complex-valued coefficients

$$z'(t) = a(t)z(t) + b(t)\bar{z}(t) + A(t)z(\theta(t)) + B(t)\bar{z}(\theta(t)) + g(t, z(t), z(\theta(t))),$$

where

$$\begin{aligned} a(t) &= \frac{1}{2}(a_{11}(t) + a_{22}(t)) + \frac{i}{2}(a_{21}(t) - a_{12}(t)), \\ b(t) &= \frac{1}{2}(a_{11}(t) - a_{22}(t)) + \frac{i}{2}(a_{21}(t) + a_{12}(t)), \end{aligned}$$

$$
\begin{aligned}
A(t) &= \frac{1}{2}(b_{11}(t)+b_{22}(t))+\frac{i}{2}(b_{21}(t)-b_{12}(t)),\\
B(t) &= \frac{1}{2}(b_{11}(t)-b_{22}(t))+\frac{i}{2}(b_{21}(t)+b_{12}(t)),\\
g(t,z,w) &= h_1\left(t,\frac{1}{2}(z+\bar z),\frac{1}{2i}(z-\bar z),\frac{1}{2}(w+\bar w),\frac{1}{2i}(w-\bar w)\right)\\
&\qquad +ih_2\left(t,\frac{1}{2}(z+\bar z),\frac{1}{2i}(z-\bar z),\frac{1}{2}(w+\bar w),\frac{1}{2i}(w-\bar w)\right).
\end{aligned}
$$

Conversely, putting $a_{11}(t)=\mathrm{Re}[a(t)+b(t)]$, $a_{12}(t)=\mathrm{Im}[b(t)-a(t)]$, $a_{21}(t)=\mathrm{Im}[a(t)+b(t)]$, $a_{22}(t)=\mathrm{Re}[a(t)-b(t)]$, $b_{11}(t)=\mathrm{Re}[A(t)+B(t)]$, $b_{12}(t)=\mathrm{Im}[B(t)-A(t)]$, $b_{21}(t)=\mathrm{Im}[A(t)+B(t)]$, $b_{22}(t)=\mathrm{Re}[A(t)-B(t)]$, $h_1(t,x,y)=\mathrm{Re}\,g(t,x_1+ix_2,y_1+iy_2)$, $h_2(t,x,y)=\mathrm{Im}\,g(t,x_1+ix_2,y_1+iy_2)$, $\mathsf{A}(t)=(a_{ij}(t))$, $\mathsf{B}(t)=(b_{ij}(t))$, the equation (1) can be written in the real form (0).

We shall use the following notation:

$\mathbb{R}$	set of all real numbers,
$\mathbb{R}_+$	set of all positive real numbers,
$\mathbb{R}_+^0$	set of all non-negative real numbers,
$\mathbb{R}_-$	set of all negative real numbers,
$\mathbb{R}_-^0$	set of all non-positive real numbers,
$\mathbb{C}$	set of all complex numbers,
$\mathcal{C}$	class of all continuous functions $[-r,0]\to\mathbb{C}$,
$AC_{\mathrm{loc}}(I,M)$	class of all locally absolutely continuous functions $I\to M$,
$L_{\mathrm{loc}}(I,M)$	class of all locally Lebesgue integrable functions $I\to M$,
$K(I\times\Omega,M)$	class of all functions $I\times\Omega\to M$ satisfying Carathéodory conditions on $I\times\Omega$,
$\mathrm{Re}\,z$	real part of z,
$\mathrm{Im}\,z$	imaginary part of z,
$\bar z$	complex conjugate of z.

2 Results

Consider the equation

$$z'(t)=a(t)z(t)+b(t)\bar z(t)+A(t)z(\theta(t))+B(t)\bar z(\theta(t))+g(t,z(t),z(\theta(t))), \quad (1)$$

where $\theta\in AC_{loc}(J,\mathbb{R})$, $a,b\in AC_{loc}(J,\mathbb{C})$, $A,B\in L_{loc}(J,\mathbb{C})$, $g\in K(J\times\mathbb{C}^2,\mathbb{C})$, $J=[t_0,\infty)$. Hereafter we shall suppose that (1) satisfies the uniqueness property of solutions. The equation (1) can be written in the form

$$z'=F(t,z_t), \quad (1')$$

where $F:J\times\mathcal{C}\to\mathbb{C}$ is defined by

$$
\begin{aligned}
F(t,\psi)=a(t)\psi(0)+b(t)\bar\psi(0)+A(t)\psi(\theta(t)-t)+B(t)\bar\psi(\theta(t)-t)\\
+g(t,\psi(0),\psi(\theta(t)-t))
\end{aligned}
$$

and z_t is the element of $\mathcal{C}$ defined by a relation $z_t(\tilde\theta) = z(t+\tilde\theta)$, $\tilde\theta \in [-r, 0]$. Instead of the case $\liminf_{t\to\infty}(|a(t)| - |b(t)|) > 0$ investigated in [10], we shall consider a case

$$\liminf_{t\to\infty}(|\operatorname{Im} a(t)| - |b(t)|) > 0, \qquad t - r \le \theta(t) \le t \ \text{ for } \ t \ge t_0 + r,$$

where $r > 0$ is a constant. Our assumptions imply the existence of numbers $T \ge t_0 + r$ and $\mu > 0$ such that

$$|\operatorname{Im} a(t)| > |b(t)| + \mu \ \text{ for } t \ge T - r, \qquad t \ge \theta(t) \ge t - r \ \text{ for } t \ge T. \tag{2}$$

Denote

$$\tilde\gamma(t) = \operatorname{Im} a(t) + \sqrt{(\operatorname{Im} a(t))^2 - |b(t)|^2}\operatorname{sgn}(\operatorname{Im} a(t)), \quad \tilde c(t) = -ib(t). \tag{3}$$

Since $|\tilde\gamma(t)| > |\operatorname{Im} a(t)|$ and $|\tilde c(t)| = |b(t)|$, the inequality

$$|\tilde\gamma(t)| > |\tilde c(t)| + \mu \tag{4}$$

is valid for $t \ge T - r$. It can be easily verified that $\tilde\gamma, \tilde c \in AC_{loc}([T-r,\infty), \mathbb{C})$. Notice that, instead of the function γ from [10], the above defined function $\tilde\gamma$ need not be positive. A simple example following Theorem 1 shows that, in some cases, our results can be applicable more often than those given in [10].

Throughout the paper we shall denote

$$\tilde\vartheta(t) = \frac{\operatorname{Re}(\tilde\gamma(t)\tilde\gamma'(t) - \bar{\tilde c}(t)\tilde c'(t)) - |\tilde\gamma(t)\tilde c'(t) - \tilde\gamma'(t)\tilde c(t)|}{\tilde\gamma^2(t) - |\tilde c(t)|^2}. \tag{5}$$

The equation (1) will be studied subject to suitable subsets of the following assumptions:

(i) The numbers $T \ge t_0 + r$ and $\mu > 0$ are such that (2) holds.

(ii) There exist functions $\tilde\varkappa, \tilde\kappa, \varrho : [T,\infty) \to \mathbb{R}$ such that

$$|\tilde\gamma(t)g(t,z,w) + \tilde c(t)\bar g(t,z,w)| \le \tilde\varkappa(t)|\tilde\gamma(t)z + \tilde c(t)\bar z| + \tilde\kappa(t)|\tilde\gamma(\theta(t))w + \tilde c(\theta(t))\bar w| + \varrho(t)$$

for $t \ge T$, $z, w \in \mathbb{C}$, where ϱ is continuous on $[T,\infty)$.

($\mathrm{ii_n}$) There exist numbers $R_n \ge 0$ and functions $\tilde\varkappa_n, \tilde\kappa_n : [T,\infty) \to \mathbb{R}$ such that

$$|\tilde\gamma(t)g(t,z,w) + \tilde c(t)\bar g(t,z,w)| \le \tilde\varkappa_n(t)|\tilde\gamma(t)z + \tilde c(t)\bar z| + \tilde\kappa_n(t)|\tilde\gamma(\theta(t))w + \tilde c(\theta(t))\bar w|$$

for $t \ge \tau_n \ge T$, $|z| > R_n$, $|w| > R_n$.

(iii) $\tilde\beta \in AC_{\mathrm{loc}}([T,\infty), \mathbb{R}^0_-)$ is a function satisfying

$$\theta'(t)\tilde\beta(t) \le -\tilde\lambda(t) \quad \text{a. e. on } [T,\infty), \tag{6}$$

where $\tilde\lambda$ is defined for $t \ge T$ by

$$\tilde\lambda(t) = \tilde\kappa(t) + (|A(t)| + |B(t)|)\frac{|\tilde\gamma(t)| + |\tilde c(t)|}{|\tilde\gamma(\theta(t))| - |\tilde c(\theta(t))|}. \tag{7}$$

(iii$_{\mathrm{n}}$) $\tilde{\beta}_n \in AC_{\mathrm{loc}}[T,\infty), \mathbb{R}^0_-)$ is a function satisfying

$$\theta'(t)\tilde{\beta}_n(t) \le -\tilde{\lambda}_n(t) \quad \text{a. e. on } [\tau_n,\infty), \tag{8}$$

where $\tilde{\lambda}_n$ is defined for $t \ge T$ by

$$\tilde{\lambda}_n(t) = \tilde{\kappa}_n(t) + (|A(t)| + |B(t)|)\frac{|\tilde{\gamma}(t)| + |\tilde{c}(t)|}{|\tilde{\gamma}(\theta(t))| - |\tilde{c}(\theta(t))|}. \tag{9}$$

(iv$_{\mathrm{n}}$) $\tilde{\Lambda}_n$ is a real locally Lebesgue integrable function satisfying the inequalities $\tilde{\beta}_n'(t) \ge \tilde{\Lambda}_n(t)\tilde{\beta}_n(t)$, $\tilde{\Theta}_n(t) \ge \tilde{\Lambda}_n(t)$ for almost all $t \in [\tau_n,\infty)$, where $\tilde{\Theta}_n$ is defined by

$$\tilde{\Theta}_n(t) = \operatorname{Re} a(t) + \tilde{\vartheta}(t) - \tilde{\varkappa}_n(t) + \tilde{\beta}_n(t). \tag{10}$$

Obviously, if A, B, $\tilde{\kappa}$, θ' are locally absolutely continuous on $[T,\infty)$ and $\tilde{\lambda}(t) \ge 0$, $\theta'(t) > 0$, the choice $\tilde{\beta}(t) = -\tilde{\lambda}(t)(\theta'(t))^{-1}$ is admissible in (iii). Similarly, if A, B, $\tilde{\kappa}_n$, θ' are locally absolutely continuous on $[T,\infty)$ and $\tilde{\lambda}_n(t) \ge 0$, $\theta'(t) > 0$, the choice $\tilde{\beta}_n(t) = -\tilde{\lambda}_n(t)(\theta'(t))^{-1}$ is admissible in (iii$_{\mathrm{n}}$).

Denote

$$\tilde{\Theta}(t) = \operatorname{Re} a(t) + \tilde{\vartheta}(t) - \tilde{\varkappa}(t). \tag{11}$$

From the assumption (i) it follows that

$$\begin{aligned}|\tilde{\vartheta}| &\le \frac{|\operatorname{Re}(\tilde{\gamma}\tilde{\gamma}' - \bar{\tilde{c}}\tilde{c}')| + |\tilde{\gamma}c' - \tilde{\gamma}'c|}{\tilde{\gamma}^2 - |\tilde{c}|^2} \le \frac{(|\tilde{\gamma}'| + |\tilde{c}'|)(|\tilde{\gamma}| + |\tilde{c}|)}{\tilde{\gamma}^2 - |\tilde{c}|^2} \\ &= \frac{|\tilde{\gamma}'| + |\tilde{c}'|}{|\tilde{\gamma}| - |\tilde{c}|} \le \frac{1}{\mu}(|\tilde{\gamma}'| + |\tilde{c}'|),\end{aligned}$$

therefore the function $\tilde{\vartheta}$ is locally Lebesgue integrable on $[T,\infty)$, assuming that (i) holds true. If relations $\tilde{\beta}_n \in AC_{\mathrm{loc}}([T,\infty),\mathbb{R}_-)$, $\tilde{\varkappa}_n \in L_{\mathrm{loc}}([T,\infty),\mathbb{R})$ and $\tilde{\beta}_n'(t)/\tilde{\beta}_n(t) \le \tilde{\Theta}_n(t)$ for almost all $t \ge \tau_n$ together with the conditions (i), (ii$_{\mathrm{n}}$) are fulfilled, then we can choose $\tilde{\Lambda}_n(t) = \tilde{\Theta}_n(t)$ for $t \in [T,\infty)$ in (iv$_{\mathrm{n}}$).

In the proof of Theorem 1 below, the following Lemma 1 will be utilized. Its proof is analogous to that of Lemma in [18], p. 131.

Lemma 1. *Let a_1, a_2, b_1, $b_2 \in \mathbb{C}$, $|a_2| > |b_2|$. Then*

$$\operatorname{Re}\frac{a_1 z + b_1\bar{z}}{a_2 z + b_2\bar{z}} \ge \frac{\operatorname{Re}(a_1\bar{a}_2 - b_1\bar{b}_2) - |a_1 b_2 - a_2 b_1|}{|a_2|^2 - |b_2|^2}$$

for $z \in \mathbb{C}$, $z \ne 0$.

Theorem 1. *Let the assumptions (i), (ii$_0$), (iii$_0$), (iv$_0$) be fulfilled for some $\tau_0 \ge T$. Suppose there exist $t_1 \ge \tau_0$ and $\nu \in (-\infty,\infty)$ such that*

$$\inf_{t \ge t_1}\left[\int_{t_1}^{t}\tilde{\Lambda}_0(s)\,ds - \ln(|\tilde{\gamma}(t)| + |\tilde{c}(t)|)\right] \ge \nu. \tag{12}$$

If $z(t)$ is any solution of (1) satisfying

$$\min_{\theta(t_1) \le s \le t_1}|z(s)| > R_0, \qquad \Delta(t_1) > R_0 e^{-\nu}, \tag{13}$$

where

$$\varDelta(t) = (|\tilde\gamma(t)| - |\tilde c(t)|)|z(t)| + \tilde\beta_0(t) \max_{\theta(t_1)\le s\le t} |z(s)| \int_{\theta(t_1)}^{t_1} (|\tilde\gamma(s)| + |\tilde c(s)|)\, ds,$$

then

$$|z(t)| \ge \frac{\varDelta(t_1)}{|\tilde\gamma(t)| + |\tilde c(t)|} \exp\left[\int_{t_1}^{t} \tilde\varLambda_0(s)\, ds\right] \tag{14}$$

for all $t \ge t_1$ for which $z(t)$ is defined.

Proof. Let $z(t)$ be any solution of (1) satisfying (13). Consider a function

$$V(t) = U(t) + \tilde\beta_0(t) \int_{\theta(t)}^{t} U(s)\, ds, \tag{15}$$

where

$$U(t) = |\tilde\gamma(t)z(t) + \tilde c(t)\bar z(t)|. \tag{16}$$

For brevity we shall denote $w(t) = z(\theta(t))$ and we shall write the function of variable t simply without indicating the variable t, for example, $\tilde\gamma$ instead of $\tilde\gamma(t)$.

In view of (15) we have

$$V' = U' + \tilde\beta_0' \int_{\theta(t)}^{t} U(s)\, ds + \tilde\beta_0 |\tilde\gamma z + \tilde c \bar z| - \tilde\beta_0 |\tilde\gamma(\theta(t))w + \tilde c(\theta(t))\bar w|\theta' \tag{17}$$

for almost all $t \ge t_1$ for which $z(t)$ is defined and $U'(t)$ exists. Put $\mathcal{K} = \{t \ge t_1 : z(t) \text{ exists}, |z(t)| > R_0\}$. Clearly $U(t) \ne 0$ for $t \in \mathcal{K}$. The derivative $U'(t)$ exists for almost all $t \in \mathcal{K}$.

Since $z(t)$ is a solution of (1), we obtain

$$\begin{aligned} UU' &= \mathrm{Re}[(\tilde\gamma\bar z + \bar{\tilde c} z)(\tilde\gamma' z + \tilde\gamma z' + \tilde c' \bar z + \tilde c \bar z')] \\ &= \mathrm{Re}\Big\{(\tilde\gamma\bar z + \bar{\tilde c} z)\Big[\tilde\gamma' z + \tilde c' \bar z + \tilde\gamma\big(az + b\bar z + Aw + B\bar w + g\big) \\ &\quad + \tilde c\big(\bar a\bar z + \bar b z + \bar A\bar w + \bar B w + \bar g\big)\Big]\Big\} \\ &= \mathrm{Re}\Big\{(\tilde\gamma\bar z + \bar{\tilde c} z)\Big[\tilde\gamma' z + \tilde c' \bar z + (\tilde\gamma a + \tilde c\bar b)z + (\tilde\gamma b + \tilde c\bar a)\bar z + \tilde\gamma\big(Aw + B\bar w + g\big) \\ &\quad + \tilde c\big(\bar A\bar w + \bar B w + \bar g\big)\Big]\Big\} \end{aligned}$$

for almost all $t \in \mathcal{K}$.

Taking into account

$$(\tilde\gamma a + \widetilde{c\bar b})\tilde c = (\tilde\gamma b + \widetilde{c\bar a})\tilde\gamma, \tag{18}$$

we get

$$
\begin{aligned}
UU' \geq{} & \operatorname{Re}\{(\tilde{\gamma}\bar{z} + \bar{\tilde{c}}z)(\tilde{\gamma}'z + \tilde{c}'\bar{z})\} + \operatorname{Re}\Big\{(\tilde{\gamma}\bar{z} + \bar{\tilde{c}}z)(\tilde{\gamma}a + \tilde{c}\bar{b})\Big(z + \frac{\tilde{c}}{\tilde{\gamma}}\bar{z}\Big)\Big\} + \\
& + \operatorname{Re}\Big\{(\tilde{\gamma}\bar{z} + \bar{\tilde{c}}z)\big(\tilde{\gamma}(Aw + B\bar{w}) + \tilde{c}(\bar{A}\bar{w} + \bar{B}w)\big)\Big\} + \\
& + \operatorname{Re}\{(\tilde{\gamma}\bar{z} + \bar{\tilde{c}}z)(\tilde{\gamma}g + \tilde{c}\bar{g})\} \\
\geq{} & U^2 \operatorname{Re}\Big(a + \frac{\tilde{c}}{\tilde{\gamma}}\bar{b}\Big) - U|Aw + B\bar{w}|(|\tilde{\gamma}| + |\tilde{c}|) - U|\tilde{\gamma}g + \tilde{c}\bar{g}| + U^2 \operatorname{Re}\frac{\tilde{\gamma}'z + \tilde{c}'\bar{z}}{\tilde{\gamma}z + \tilde{c}\bar{z}}.
\end{aligned}
$$

By the use of Lemma 1 we get

$$
\operatorname{Re}\frac{\tilde{\gamma}'z + \tilde{c}'\bar{z}}{\tilde{\gamma}z + \tilde{c}\bar{z}} \geq \tilde{\vartheta}.
$$

The last inequality together with (9), taken for $n = 0$, the assumption (ii_0) and the relation

$$
\operatorname{Re}\Big(a + \frac{\tilde{c}}{\tilde{\gamma}}\bar{b}\Big) = \operatorname{Re} a \tag{19}
$$

yield

$$
\begin{aligned}
UU' \geq{} & U^2(\operatorname{Re} a + \tilde{\vartheta} - \tilde{\varkappa}_0) - U(|A| + |B|)|w|(|\tilde{\gamma}| + |\tilde{c}|) \\
& \qquad - U\tilde{\kappa}_0|\tilde{\gamma}(\theta(t))w + \tilde{c}(\theta(t))\bar{w}| \\
\geq{} & U^2(\operatorname{Re} a + \tilde{\vartheta} - \tilde{\varkappa}_0) - U\tilde{\lambda}_0|\tilde{\gamma}(\theta(t))w + \tilde{c}(\theta(t))\bar{w}|.
\end{aligned}
$$

Therefore

$$
U' \geq U(\operatorname{Re} a + \tilde{\vartheta} - \tilde{\varkappa}_0) - \tilde{\lambda}_0|\tilde{\gamma}(\theta(t))w + \tilde{c}(\theta(t))\bar{w}| \tag{20}
$$

for almost all $t \in \mathcal{K}$. The relation (17) together with the inequality (20) gives

$$
V' \geq U(\operatorname{Re} a + \tilde{\vartheta} - \tilde{\varkappa}_0 + \tilde{\beta}_0) - |\tilde{\gamma}(\theta(t))w + \tilde{c}(\theta(t))\bar{w}|(\tilde{\lambda}_0 + \tilde{\beta}_0\theta') + \tilde{\beta}_0' \int_{\theta(t)}^{t} U(s)\,ds.
$$

Using (8) and (10) for $n = 0$, we obtain

$$
V'(t) \geq U(t)\tilde{\Theta}_0(t) + \tilde{\beta}_0'(t) \int_{\theta(t)}^{t} U(s)\,ds.
$$

Hence, in view of (iv_0)

$$
V'(t) - \tilde{\Lambda}_0(t)V(t) \geq 0 \tag{21}
$$

for almost all $t \in \mathcal{K}$. Multiplying (21) by $\exp\Big[-\int_{t_1}^{t} \tilde{\Lambda}_0(s)\,ds\Big]$ and integrating over $[t_1, t]$, we get

$$
V(t)\exp\Big[-\int_{t_1}^{t} \tilde{\Lambda}_0(s)\,ds\Big] - V(t_1) \geq 0
$$

on any interval $[t_1, \omega)$ where the solution $z(t)$ exists and satisfies the inequality $|z(t)| > R_0$. Now, with respect to (15), (16) and $\tilde{\beta}_0 \le 0$, we have

$$(|\tilde{\gamma}(t)|+|\tilde{c}(t)|)|z(t)| \ge V(t) \ge V(t_1)\exp\left[\int_{t_1}^{t} \tilde{\Lambda}_0(s)\,ds\right] \ge \Delta(t_1)\exp\left[\int_{t_1}^{t} \tilde{\Lambda}_0(s)\,ds\right].$$

If (13) is fulfilled, there is a $R > R_0$ such that $\Delta(t_1) > Re^{-\nu}$. By virtue of (12) and (13) we can easily see that

$$|z(t)| \ge \frac{\Delta(t_1)}{|\tilde{\gamma}(t)| + |\tilde{c}(t)|}\exp\left[\int_{t_1}^{t} \tilde{\Lambda}_0(s)\,ds\right] \ge Re^{-\nu}e^{\nu} = R$$

for all $t \ge t_1$ for which $z(t)$ is defined. □

In the next example we give an equation of the form (1) to which Theorem 1 of [10] is not applicable, but Theorem 1 of the present paper can be applied.

Example 1. Consider the equation (1) where $a(t) \equiv 8+6i$, $b(t) \equiv 5$, $A(t) \equiv 0$, $B(t) \equiv 0$, $\theta(t) = t + \frac{1}{2}(\sin t - 1)$, $g(t,z,w) = 6z + \mathrm{e}^{-t}\,w$. Obviously $t - 1 \le \theta(t) \le t$ and $\frac{1}{2} \le \theta'(t) \le \frac{3}{2}$. Suppose $t_0 = 1$ and $T \ge 2$. Then $\gamma(t) = |a(t)| + \sqrt{|a(t)|^2 - |b(t)|^2} \equiv 10+5\sqrt{3}$, $c(t) = \bar{a}(t)b(t)/|a(t)| \equiv 4-3i$, $\tilde{\gamma} \equiv 6+\sqrt{11}$, $\tilde{c} \equiv -5i$. Further,

$$|\gamma(t)g(t,z,w) + c(t)\bar{g}(t,z,w)| \le 6|\gamma(t)z + c(t)\bar{z}| + \mathrm{e}^{-t}|\gamma(\theta(t))w + c(\theta(t))\bar{w}|,$$
$$|\tilde{\gamma}(t)g(t,z,w) + \tilde{c}(t)\bar{g}(t,z,w)| \le 6|\tilde{\gamma}(t)z + \tilde{c}(t)\bar{z}| + \mathrm{e}^{-t}|\tilde{\gamma}(\theta(t))w + \tilde{c}(\theta(t))\bar{w}|.$$

Following Theorem 1 of [10] we obtain $\varkappa_0(t) \equiv 6$, $\kappa_0(t) = \mathrm{e}^{-t}$, $\vartheta(t) \equiv 0$, $\alpha(t) \equiv \frac{1}{2}$, $\Lambda_0(t) \le \Theta_0(t) = -2 + \beta_0(t) \le -2 - \lambda_0(t)(\theta'(t))^{-1} \le -2 < 0$ and we see that Theorem 1 of [10] is not applicable, because the relation (12) in [10] cannot be fulfilled. On the other hand, taking $\tilde{\varkappa}_0(t) \equiv 6$, $\tilde{\kappa}_0(t) = \mathrm{e}^{-t}$, $\tau_0 = T$, $R_0 = 0$, $\tilde{\vartheta}(t) \equiv 0$, $\tilde{\beta}_0(t) = -2\,\mathrm{e}^{-t}$, $\tilde{\Lambda}_0(t) = \tilde{\Theta}_0(t) = 2 - 2\,\mathrm{e}^{-t}$ (> 0) in Theorem 1 of the present paper, we have $\theta'(t)\tilde{\beta}_0(t) \le -\tilde{\lambda}_0(t)$, $\tilde{\beta}_0'(t) \ge \tilde{\Theta}_0(t)\tilde{\beta}_0(t)$ for $t \in [T, \infty)$ and Theorem 1 is applicable to the considered equation.

Remark 1. Putting $\theta(t) \equiv t - r_0$, where $0 \le r_0 \le r$, in Theorem 1, we obtain a slight generalization of Theorem 1 of [9]. Notice that in the case $r_0 = 0$ (i. e. $\theta(t) \equiv t$) the condition (13) takes the form

$$|z(t_1)| > R_0 \max\left\{1, \frac{1}{(|\tilde{\gamma}(t_1)| - |\tilde{c}(t_1)|)e^{\nu}}\right\}.$$

Corollary 1. *Let the assumptions of Theorem 1 be fulfilled with* $R_0 > 0$. *If*

$$\liminf_{t\to\infty}\left[\int_{t_1}^{t} \tilde{\Lambda}_0(s)\,ds - \ln(|\tilde{\gamma}(t)| + |\tilde{c}(t)|)\right] = \varsigma > \nu, \tag{22}$$

then to any ε, $0 < \varepsilon < R_0 e^{\varsigma-\nu}$, *there is a* $t_2 \ge t_1$ *such that*

$$|z(t)| > \varepsilon \tag{23}$$

for all $t \ge t_2$ *for which* $z(t)$ *is defined.*

Proof. Without loss of generality we can assume $\varepsilon > R_0$. Choose χ, $0 < \chi < 1$ such that $R_0 < \varepsilon < \chi R_0 e^{\varsigma - \nu}$. In view of (22) there is $t_2 \geq t_1$ such that

$$\int_{t_1}^{t} \tilde{\varLambda}_0(s)\, ds - \ln(|\tilde{\gamma}(t)| + |\tilde{c}(t)|) > \varsigma + \ln \chi$$

for $t \geq t_2$. Hence

$$\int_{t_1}^{t} \tilde{\varLambda}_0(s)\, ds - \ln(|\tilde{\gamma}(t)| + |\tilde{c}(t)|) > \nu + \ln \frac{\varepsilon}{R_0}$$

for $t \geq t_2$. The estimation (14) together with (13) now yields

$$|z(t)| > R_0 e^{-\nu} e^{\nu} \frac{\varepsilon}{R_0} = \varepsilon$$

for all $t \geq t_2$ for which $z(t)$ is defined. $\square$

Corollary 2. *Let the assumptions of Theorem 1 be fulfilled with $R_0 > 0$. If*

$$\lim_{t\to\infty} \left[\int_{t_1}^{t} \tilde{\varLambda}_0(s)\, ds - \ln(|\tilde{\gamma}(t)| + |\tilde{c}(t)|) \right] = \infty,$$

then for any $\varepsilon > 0$ there exists a $t_2 \geq t_1$ such that (23) *holds for all $t \geq t_2$ for which $z(t)$ is defined.*

In the proof of the following theorem we shall utilize Ważewski topological principle for retarded functional differential equations of Carathéodory type. For details of this theory see results of K. P. Rybakowski [21].

Theorem 2. *Let the conditions (i), (ii), (iii) be fulfilled and $\tilde{\varLambda}$, θ' be continuous functions such that the inequality $\tilde{\varLambda}(t) \leq \tilde{\varTheta}(t)$ holds a. e. on $[T, \infty)$, where $\tilde{\varTheta}$ is defined by* (11). *Suppose that $\xi : [T - r, \infty) \to \mathbb{R}$ is a continuous function such that*

$$\tilde{\varLambda}(t) + \tilde{\beta}(t)\theta'(t) \exp\left[-\int_{\theta(t)}^{t} \xi(s)\, ds \right] - \xi(t) > \varrho(t) C^{-1} \exp\left[-\int_{T}^{t} \xi(s)\, ds \right] \quad (24)$$

for $t \in [T, \infty]$ and some constant $C > 0$. Then there exists a $t_2 > T$ and a solution $z_0(t)$ of (1) *satisfying*

$$|z_0(t)| \leq \frac{C}{|\tilde{\gamma}(t)| - |\tilde{c}(t)|} \exp\left[\int_{T}^{t} \xi(s)\, ds \right] \quad (25)$$

for $t \geq t_2$.

Proof. Rewrite the equation (1) in the form (1′). Let $\tau > T$. Put

$$\begin{aligned}
\widetilde{U}(t, z, \bar{z}) &= |\tilde{\gamma}(t) z + \tilde{c}(t)\bar{z}| - \varphi(t),\\
\varphi(t) &= C \exp\left[\int_{T}^{t} \xi(s)\, ds \right],\\
\varOmega^0 &= \{(t, z) \in (\tau, \infty) \times \mathbb{C} : \widetilde{U}(t, z, \bar{z}) < 0\},\\
\varOmega_{\widetilde{U}} &= \{(t, z) \in (\tau, \infty) \times \mathbb{C} : \widetilde{U}(t, z, \bar{z}) = 0\}.
\end{aligned}$$

It can be easily verified that Ω^0 is a polyfacial set generated by functions $\hat{U}(t) = \tau - t$, $\widetilde{U}(t, z, \bar{z})$ (see Rybakowski [21, p. 134]). It holds that $\Omega_{\widetilde{U}} \subset \partial\Omega^0$. As $(|\tilde{\gamma}(t)| + |\tilde{c}(t)|)|z(t)| \geq |\tilde{\gamma}(t)z + \tilde{c}(t)\bar{z}|$, we have

$$|z| \geq \frac{\varphi(t)}{|\tilde{\gamma}(t)| + |\tilde{c}(t)|} = \frac{C}{|\tilde{\gamma}(t)| + |\tilde{c}(t)|} \exp\left[\int_T^t \xi(s)\, ds\right] > 0$$

for $(t, z) \in \Omega_{\widetilde{U}}$. It holds

$$D^+\hat{U}(t) = \frac{\partial}{\partial t}(\tau - t) = -1 < 0.$$

Let $(t^*, \zeta) \in \Omega_{\widetilde{U}}$ and $\phi \in \mathcal{C}$ be such that $\phi(0) = \zeta$ and $(t^* + \theta, \phi(\theta)) \in \Omega^0$ for all $\theta \in [-r, 0)$. If $(t, \psi) \in (\tau, \infty) \times \mathcal{C}$, then

$$\begin{aligned} D^+\widetilde{U}(t, \psi(0), \bar{\psi}(0)) := \limsup_{h\to 0+} (1/h)[&\widetilde{U}(t+h, \psi(0) + hF(t,\psi), \bar{\psi}(0) + h\bar{F}(t,\psi)) \\ &- \widetilde{U}(t, \psi(0), \bar{\psi}(0))] \\ = \frac{\partial \widetilde{U}(t, \psi(0), \bar{\psi}(0))}{\partial t} + \frac{\partial \widetilde{U}(t, \psi(0), \bar{\psi}(0))}{\partial z}&F(t,\psi) + \frac{\partial \widetilde{U}(t, \psi(0), \bar{\psi}(0))}{\partial \bar{z}}\bar{F}(t,\psi). \end{aligned}$$

Therefore

$$\begin{aligned} D^+\widetilde{U}(t, \psi(0), \bar{\psi}(0)) &= |\tilde{\gamma}(t)\psi(0) + \tilde{c}(t)\bar{\psi}(0)| \operatorname{Re} \frac{\tilde{\gamma}'(t)\psi(0) + \tilde{c}'(t)\bar{\psi}(0)}{\tilde{\gamma}(t)\psi(0) + \tilde{c}(t)\bar{\psi}(0)} - \varphi'(t) \\ &+ \tfrac{1}{2}|\tilde{\gamma}(t)\psi(0) + \tilde{c}(t)\bar{\psi}(0)|^{-1} \operatorname{Re}\{[\tilde{\gamma}(t)(\tilde{\gamma}(t)\bar{\psi}(0) + \bar{\tilde{c}}(t)\psi(0)) \\ &\quad + (\tilde{\gamma}(t)\psi(0) + \tilde{c}(t)\bar{\psi}(0))\bar{\tilde{c}}(t)]F(t,\psi) \\ &\quad\quad + [\tilde{c}(t)(\tilde{\gamma}(t)\bar{\psi}(0) + \bar{\tilde{c}}(t)\psi(0)) + \tilde{\gamma}(t)(\tilde{\gamma}(t)\psi(0) + \tilde{c}(t)\bar{\psi}(0))]\bar{F}(t,\psi)\} \end{aligned}$$

provided that the derivatives $\tilde{\gamma}'(t)$, $\tilde{c}'(t)$ exist and that $\psi(0) \neq 0$. Thus

$$\begin{aligned} D^+\widetilde{U}(t, \psi(0), \bar{\psi}(0)) &= |\tilde{\gamma}(t)\psi(0) + \tilde{c}(t)\bar{\psi}(0)| \operatorname{Re} \frac{\tilde{\gamma}'(t)\psi(0) + \tilde{c}'(t)\bar{\psi}(0)}{\tilde{\gamma}(t)\psi(0) + \tilde{c}(t)\bar{\psi}(0)} - \varphi'(t) \\ &+ |\tilde{\gamma}(t)\psi(0) + \tilde{c}(t)\bar{\psi}(0)|^{-1} \operatorname{Re}\{\tilde{\gamma}(t)(\tilde{\gamma}(t)\bar{\psi}(0) + \bar{\tilde{c}}(t)\psi(0))F(t,\psi) \\ &\quad + \tilde{c}(t)(\tilde{\gamma}(t)\bar{\psi}(0) + \bar{\tilde{c}}(t)\psi(0))\bar{F}(t,\psi)\} \\ = |\tilde{\gamma}(t)\psi(0) + \tilde{c}(t)\bar{\psi}(0)| &\operatorname{Re} \frac{\tilde{\gamma}'(t)\psi(0) + \tilde{c}'(t)\bar{\psi}(0)}{\tilde{\gamma}(t)\psi(0) + \tilde{c}(t)\bar{\psi}(0)} - \varphi'(t) \\ + |\tilde{\gamma}(t)\psi(0) + \tilde{c}(t)\bar{\psi}(0)|^{-1} &\operatorname{Re}\{(\tilde{\gamma}(t)\bar{\psi}(0) + \bar{\tilde{c}}(t)\psi(0))(\tilde{\gamma}(t)F(t,\psi) + \tilde{c}(t)\bar{F}(t,\psi))\}. \end{aligned}$$

Using (18), (19) and (ii), similarly to the proof of Theorem 1, we obtain

$$\begin{aligned} &D^+\widetilde{U}(t, \psi(0), \bar{\psi}(0)) \geq |\tilde{\gamma}(t)\psi(0) + \tilde{c}(t)\bar{\psi}(0)| \operatorname{Re} a(t) \\ &- |A(t)\psi(\theta(t) - t) + B(t)\bar{\psi}(\theta(t) - t)|(|\tilde{\gamma}(t)| + |\tilde{c}(t)|) - \tilde{\varkappa}(t)|\tilde{\gamma}(t)\psi(0) + \tilde{c}(t)\bar{\psi}(0)| \\ &- \tilde{\kappa}(t)|\tilde{\gamma}(\theta(t))\psi(\theta(t) - t) + \tilde{c}(\theta(t))\bar{\psi}(\theta(t) - t)| \\ &\qquad + \tilde{\vartheta}(t)|\tilde{\gamma}(t)\psi(0) + \tilde{c}(t)\bar{\psi}(0)| - \varrho(t) - \varphi'(t) \end{aligned}$$

and consequently

$$\begin{aligned}
D^+\widetilde{U}(t,\psi(0),\bar{\psi}(0)) \geq{}& (\operatorname{Re} a(t)+\tilde{\vartheta}(t)-\tilde{\varkappa}(t))|\tilde{\gamma}(t)\psi(0)+\tilde{c}(t)\bar{\psi}(0)|\\
&-\tilde{\lambda}(t)|\tilde{\gamma}(\theta(t))\psi(\theta(t)-t)+\tilde{c}(\theta(t))\bar{\psi}(\theta(t)-t)|-\varrho(t)-\varphi'(t)\geq\\
\tilde{\Theta}(t)|\tilde{\gamma}(t)\psi(0)+\tilde{c}(t)\bar{\psi}(0)|&+\tilde{\beta}(t)\theta'(t)|\tilde{\gamma}(\theta(t))\psi(\theta(t)-t)+\tilde{c}(\theta(t))\bar{\psi}(\theta(t)-t)|\\
&-\varrho(t)-\varphi'(t)\geq\\
\tilde{\Lambda}(t)|\tilde{\gamma}(t)\psi(0)+\tilde{c}(t)\bar{\psi}(0)|&+\tilde{\beta}(t)\theta'(t)|\tilde{\gamma}(\theta(t))\psi(\theta(t)-t)+\tilde{c}(\theta(t))\bar{\psi}(\theta(t)-t)|\\
&-\varrho(t)-\varphi'(t)
\end{aligned}$$

for almost all $t\in(\tau,\infty)$ and for $\psi\in\mathcal{C}$ sufficiently close to ϕ. Replacing t and ψ by t^* and ϕ, respectively, in the last expression, we get

$$\begin{aligned}
&\tilde{\Lambda}(t^*)|\tilde{\gamma}(t^*)\phi(0)+\tilde{c}(t^*)\bar{\phi}(0)|\\
&\quad+\tilde{\beta}(t^*)\theta'(t^*)|\tilde{\gamma}(\theta(t^*))\phi(\theta(t^*)-t^*)+\tilde{c}(\theta(t^*))\bar{\phi}(\theta(t^*)-t^*)|-\varrho(t^*)-\varphi'(t^*)\\
&\geq\tilde{\Lambda}(t^*)|\tilde{\gamma}(t^*)\zeta+\tilde{c}(t^*)\bar{\zeta}|+\tilde{\beta}(t^*)\theta'(t^*)\varphi(\theta(t^*))-\varrho(t^*)-\varphi'(t^*)\\
&\geq\tilde{\Lambda}(t^*)\varphi(t^*)+\tilde{\beta}(t^*)\theta'(t^*)\varphi(\theta(t^*))-\varrho(t^*)-\varphi'(t^*)\\
&=\tilde{\Lambda}(t^*)C\exp\left[\int_T^{t^*}\xi(s)\,ds\right]+\tilde{\beta}(t^*)\theta'(t^*)C\exp\left[\int_T^{\theta(t^*)}\xi(s)\,ds\right]\\
&\qquad\qquad-\varrho(t^*)-C\xi(t^*)\exp\left[\int_T^{t^*}\xi(s)\,ds\right]\\
&=\left\{\tilde{\Lambda}(t^*)+\tilde{\beta}(t^*)\theta'(t^*)\exp\left[-\int_{\theta(t^*)}^{t^*}\xi(s)\,ds\right]-\xi(t^*)\right\}C\exp\left[\int_T^{t^*}\xi(s)\,ds\right]\\
&\quad-\varrho(t^*)>0.
\end{aligned}$$

Therefore, in view of the continuity, $D^+\widetilde{U}(t,\psi(0),\bar{\psi}(0))>0$ holds for ψ sufficiently close to ϕ and almost all t sufficiently close to t^*. Hence Ω^0 is a regular polyfacial set with respect to $(1')$.

Choose $Z=\{(t_2,z)\in\Omega^0\cup\Omega_{\widetilde{U}}\}$, where $t_2>\tau+r$ is fixed. It can be easily verified that $Z\cap\Omega_{\widetilde{U}}$ is a retract of $\Omega_{\widetilde{U}}$, but $Z\cap\Omega_{\widetilde{U}}$ is not a retract of Z. Let $\eta\in\mathcal{C}$ be such that $\eta(0)=1$ and $0\leq\eta(\theta)<1$ for $\theta\in[-r,0)$. Define a mapping $p:Z\to\mathcal{C}$ for $(t_2,z)\in Z$ by the relation

$$\begin{aligned}
p(t_2,z)(\theta)=\frac{\varphi(t_2+\theta)\eta(\theta)}{(\tilde{\gamma}^2(t_2+\theta)-|\tilde{c}(t_2+\theta)|^2)\varphi(t_2)}&[(\tilde{\gamma}(t_2)\tilde{\gamma}(t_2+\theta)-\bar{\tilde{c}}(t_2)\tilde{c}(t_2+\theta))z\\
&+(\tilde{\gamma}(t_2+\theta)\tilde{c}(t_2)-\tilde{\gamma}(t_2)\tilde{c}(t_2+\theta))\bar{z}].
\end{aligned}$$

The mapping p is continuous and it holds that

$$p(t_2,z)(0)=z\ \text{ for } (t_2,z)\in Z,\qquad p(t_2,0)(\theta)=0\ \text{ for } \theta\in[-r,0].$$

Since

$$\tilde{\gamma}(t_2+\theta)p(t_2,z)(\theta)+\tilde{c}(t_2+\theta)\overline{p(t_2,z)(\theta)}=\frac{\varphi(t_2+\theta)\eta(\theta)}{\varphi(t_2)}(\tilde{\gamma}(t_2)z+\tilde{c}(t_2)\bar{z}),$$

we have

$$|\tilde{\gamma}(t_2)z + \tilde{c}(t_2)\bar{z}| < \varphi(t_2)$$

and

$$|\tilde{\gamma}(t_2+\theta)p(t_2,z)(\theta) + \tilde{c}(t_2+\theta)\overline{p(t_2,z)(\theta)}| < \varphi(t_2+\theta) \tag{26}$$

for $(t_2, z) \in Z \cap \Omega^0$ and $\theta \in [-r, 0]$. Clearly, the inequality (26) holds also for $(t_2, z) \in Z \cap \Omega_{\tilde{U}}$ and $\theta \in [-r, 0)$.

Using a topological principle for retarded functional differential equations (see Rybakowski [21, Theorem 2.1]), we infer that there is a solution $z_0(t)$ of (1) such that $(t, z_0(t)) \in \Omega^0$ for all $t \geq t_2$ for which the solution $z_0(t)$ exists. Obviously $z_0(t)$ exists for all $t \geq t_2$ and

$$(|\tilde{\gamma}(t)| - |\tilde{c}(t)|)|z_0(t)| \leq |\tilde{\gamma}(t)z_0(t) + \tilde{c}(t)\bar{z}_0(t)| \leq \varphi(t) \qquad \text{for } t \geq t_2.$$

Hence

$$|z_0(t)| \leq \frac{\varphi(t)}{|\tilde{\gamma}(t)| - |\tilde{c}(t)|} \qquad \text{for } t \geq t_2.$$

□

Remark 2. 1. If $\theta'(t) \geq 0$, $\eta_1(t)\tilde{\Lambda}(t) > \theta'(t)|\tilde{\beta}(t)| + C^{-1}\varrho(t) > 0$, where $0 < \eta_1(t) \leq 1$, the functions η_1, $\tilde{\Lambda}$ are continuous on $[T, \infty)$ and $\tilde{\Lambda}(t) \leq \tilde{\Theta}(t)$ a. e. on $[T, \infty)$, then the choice $\xi(t) = \eta_1(t)\tilde{\Lambda}(t) + \theta'(t)\tilde{\beta}(t) - C^{-1}\varrho(t)$ is possible in (24). Moreover, in some cases, the condition $\theta'(t)|\tilde{\beta}(t)| + C^{-1}\varrho(t) > 0$ can be omitted if Theorem 2 is used. For instance, the identity $\theta'(t)|\tilde{\beta}(t)| + C^{-1}\varrho(t) \equiv 0$ implies $\theta'(t)\tilde{\beta}(t) \equiv 0$, $\varrho(t) \equiv 0$ and consequently, in view of (6), (7), (ii), we have $\tilde{\lambda}(t) \equiv 0$, $\tilde{\kappa}(t) \equiv 0$, $A(t) \equiv 0$, $B(t) \equiv 0$, $g(t, 0, 0) \equiv 0$. Thus the equation (1) has the trivial solution $z_0(t) \equiv 0$ in this case.
2. Taking $\theta(t) \equiv t - r_0$, where $0 \leq r_0 \leq r$, in Theorem 2, we get a generalization of Theorem 5 of [8].

Corollary 3. *Let the assumptions of Theorem 2 be satisfied. If*

$$\limsup_{t\to\infty} \left[\frac{1}{|\tilde{\gamma}(t)| - |\tilde{c}(t)|} \exp\left(\int_T^t \xi(s)\, ds \right) \right] < \infty,$$

then there is a bounded solution $z_0(t)$ of (1)*. If*

$$\lim_{t\to\infty} \left[\frac{1}{|\tilde{\gamma}(t)| - |\tilde{c}(t)|} \exp\left(\int_T^t \xi(s)\, ds \right) \right] = 0,$$

then there is a solution $z_0(t)$ of (1) *such that*

$$\lim_{t\to\infty} z_0(t) = 0.$$

Theorem 3. *Suppose that the hypotheses (i), (ii), (ii$_n$), (iii), (iii$_n$), (iv$_n$) are fulfilled for $\tau_n \geq T$ and $n \in \mathbb{N}$, where $R_n > 0$, $\inf_{n\in\mathbb{N}} R_n = 0$. Let $\tilde{\Lambda}$ be a continuous function satisfying the inequality $\tilde{\Lambda}(t) \leq \tilde{\Theta}(t)$ a. e. on $[T, \infty)$, where*

$\tilde{\Theta}$ is defined by (11)*. Assume that $\xi : [T-r,\infty) \to \mathbb{R}$ is a continuous function such that*

$$\tilde{\Lambda}(t)+\tilde{\beta}(t)\theta'(t)\exp\left[-\int_{\theta(t)}^{t}\xi(s)\,ds\right]-\xi(t) > \varrho(t)C^{-1}\exp\left(-\int_{T}^{t}\xi(s)\,ds\right) \quad (27)$$

for $t \in [T,\infty)$ and some constant $C > 0$. Suppose

$$\limsup_{t\to\infty}\left[\int_{T}^{t}(\tilde{\Lambda}_n(s)-\xi(s))\,ds+\ln\frac{|\tilde{\gamma}(t)|-|\tilde{c}(t)|}{|\tilde{\gamma}(t)|+|\tilde{c}(t)|}\right]=\infty, \quad (28)$$

$$\lim_{t\to\infty}\left[\tilde{\beta}_n(t)\max_{\theta(t)\le s\le t}\frac{\exp\left[\int_T^s\xi(\sigma)\,d\sigma\right]}{|\tilde{\gamma}(s)|-|\tilde{c}(s)|}\int_{\theta(t)}^{t}(|\tilde{\gamma}(s)|+|\tilde{c}(s)|)\,ds\right]=0, \quad (29)$$

$$\inf_{\tau_n\le s\le t<\infty}\left[\int_{s}^{t}\tilde{\Lambda}_n(\sigma)\,d\sigma-\ln(|\tilde{\gamma}(t)|+|\tilde{c}(t)|)\right]\ge\nu \quad (30)$$

for $n \in \mathbb{N}$, where $\nu \in (-\infty,\infty)$. Then there exists a solution $z_0(t)$ of (1) *such that*

$$\lim_{t\to\infty}\min_{\theta(t)\le s\le t}|z_0(s)|=0. \quad (31)$$

Proof. By the use of Theorem 2 we observe that there is a $t_2 \ge T$ and a solution $z_0(t)$ of (1) with property

$$|z_0(t)| \le \frac{C}{|\tilde{\gamma}(t)|-|\tilde{c}(t)|}\exp\left[\int_T^t\xi(s)\,ds\right] \quad (32)$$

for $t \ge t_2$. Suppose that (31) is not satisfied. Then there is $\varepsilon_0 > 0$ such that

$$\limsup_{t\to\infty}\min_{\theta(t)\le s\le t}|z_0(s)|>\varepsilon_0.$$

Choose $N \in \mathbb{N}$ such that

$$\max\left\{R_N,\frac{2}{\mu}R_Ne^{-\nu}\right\}<\varepsilon_0.$$

It holds that

$$\min_{\theta(\tau)\le s\le\tau}|z_0(s)|>\max\left\{R_N,\frac{2}{\mu}R_Ne^{-\nu}\right\} \quad (33)$$

for some $\tau > \max\{T,\tau_N,t_2\}$. In view of (29) we can suppose that

$$|\tilde{\beta}_N(\tau)|C\max_{\theta(\tau)\le s\le\tau}\frac{\exp\left[\int_T^s\xi(\sigma)\,d\sigma\right]}{|\tilde{\gamma}(s)|-|\tilde{c}(s)|}\int_{\theta(\tau)}^{\tau}(|\tilde{\gamma}(s)|+|\tilde{c}(s)|)\,ds<\frac{1}{2}R_Ne^{-\nu}. \quad (34)$$

Therefore, taking into account (4), (32), (33), (34) and the nonpositiveness of

β_N, we have

$$
\begin{aligned}
&(|\tilde\gamma(\tau)| - |\tilde c(\tau)|)|z_0(\tau)| + \tilde\beta_N(\tau) \max_{\theta(\tau)\le s\le\tau} |z_0(s)| \int_{\theta(\tau)}^{\tau} (|\tilde\gamma(s)| + |\tilde c(s)|)\, ds \\
&\ge (|\tilde\gamma(\tau)| - |\tilde c(\tau)|)|z_0(\tau)| \\
&\qquad\qquad + \tilde\beta_N(\tau) C \max_{\theta(\tau)\le s\le\tau} \frac{\exp\left[\int_T^s \xi(\sigma)\, d\sigma\right]}{|\tilde\gamma(s)| - |\tilde c(s)|} \int_{\theta(\tau)}^{\tau} (|\tilde\gamma(s)| + |\tilde c(s)|)\, ds \\
&\ge \mu \frac{2}{\mu} R_N e^{-\nu} - \frac{1}{2} R_N e^{-\nu} > R_N e^{-\nu}.
\end{aligned}
$$

Moreover (30) implies

$$
\inf_{\tau\le t<\infty} \left[\int_\tau^t \tilde\Lambda_N(s)\, ds - \ln(|\tilde\gamma(t)| + |\tilde c(t)|) \right] \ge \nu > -\infty.
$$

By Theorem 1 we obtain an estimation

$$
|z_0(t)| \ge \frac{\Psi(\tau)}{|\tilde\gamma(t)| + |\tilde c(t)|} \exp\left[\int_\tau^t \tilde\Lambda_N(s)\, ds \right] \tag{35}
$$

for all $t \ge \tau$, Ψ being defined by

$$
\Psi(\tau) = (|\tilde\gamma(\tau)| - |\tilde c(\tau)|)|z_0(\tau)| + \tilde\beta_N(\tau) \max_{\theta(\tau)\le s\le\tau} |z_0(s)| \int_{\theta(\tau)}^{\tau} (|\tilde\gamma(s)| + |\tilde c(s)|)\, ds.
$$

The relation (32) together with (35) yield

$$
\frac{\Psi(\tau)}{|\tilde\gamma(t)| + |\tilde c(t)|} \exp\left[\int_\tau^t \tilde\Lambda_N(s)\, ds \right] \le \frac{C}{|\tilde\gamma(t)| - |\tilde c(t)|} \exp\left[\int_T^t \xi(s)\, ds \right],
$$

i. e.

$$
\int_T^t [\tilde\Lambda_N(s) - \xi(s)]\, ds + \ln \frac{|\tilde\gamma(t)| - |\tilde c(t)|}{|\tilde\gamma(t)| + |\tilde c(t)|} \le \int_T^\tau \tilde\Lambda_N(s)\, ds - \ln[C^{-1}\Psi(\tau)]
$$

for $t \ge \tau$. However, the last inequality contradicts (28) and Theorem 3 is proved. □

Remark 3. Putting $\theta(t) \equiv t - r_0$, where $0 \le r_0 \le r$, in Theorem 3, we obtain a generalization of Theorem 3 of [9]. Notice that in the case $r_0 = 0$ (i. e. $\theta(t) \equiv t$) the condition (29) can be omitted and (31) is of the form $\lim_{t\to\infty} |z(t)| = 0$.

3 Examples

In this section we illustrate the applicability of the results by several examples. We consider a simpler equation

$$
z'(t) = a_0 z(t) + b_0 \bar z(t) + A(t) z(\theta(t)) + B(t) \bar z(\theta(t)) + g(t, z(t), z(\theta(t))), \tag{36}
$$

where $\theta \in AC_{\mathrm{loc}}(J,\mathbb{R})$, $A, B \in L_{\mathrm{loc}}(J,\mathbb{C})$, $g \in K(J \times \mathbb{C}^2,\mathbb{C})$, $J = [t_0,\infty)$, $a_0, b_0 \in \mathbb{C}$ being constants. We suppose the existence of $r > 0$ and $T \geq t_0 + r$ such that $|\operatorname{Im} a_0| > |b_0|$ and

$$t - r \leq \theta(t) \leq t \qquad \text{for } t \geq T.$$

In this case we have

$$\tilde{\gamma}(t) = \gamma := \operatorname{Im} a_0 + \sqrt{(\operatorname{Im} a_0)^2 - |b_0|^2}\,\operatorname{sgn}\operatorname{Im} a_0, \quad \tilde{c}(t) = \tilde{c} := -ib_0.$$

Clearly $\tilde{\vartheta}(t) \equiv \tilde{\vartheta} := 0$, $\mu = |\operatorname{Im} a_0| - |b_0|$.

We assume

$$|g(t,z,w)| \leq \hat{\varkappa}(t)|z| + \tilde{\kappa}(t)|w| + G(t) \qquad \text{for } t \geq T, \quad z, w \in \mathbb{C},$$

where $\hat{\varkappa} \in L_{\mathrm{loc}}([T,\infty),\mathbb{R}^0_+)$, $\hat{\kappa} \in AC_{\mathrm{loc}}([T,\infty),\mathbb{R}^0_+)$ and G is a real continuous function on $[T,\infty)$. It can be easily verified that

$$|\tilde{\gamma} g(t,z,w) + \tilde{c}\bar{g}(t,z,w)| \leq \tilde{\varkappa}(t)|\tilde{\gamma} z + \tilde{c}\bar{z}| + \tilde{\kappa}(t)|\tilde{\gamma} w + \tilde{c}\bar{w}| + \varrho(t)$$

for $t \geq T$, $z, w \in \mathbb{C}$, where

$$\tilde{\varkappa}(t) = \frac{|\tilde{\gamma}| + |\tilde{c}|}{|\tilde{\gamma}| - |\tilde{c}|}\hat{\varkappa}(t), \quad \tilde{\kappa}(t) = \frac{|\tilde{\gamma}| + |\tilde{c}|}{|\tilde{\gamma}| - |\tilde{c}|}\hat{\kappa}(t), \quad \varrho(t) = (|\tilde{\gamma}| + |\tilde{c}|)G(t).$$

Similarly, if $R_n > 0$, we have

$$|\tilde{\gamma} g(t,z,w) + \tilde{c}\bar{g}(t,z,w)| \leq \tilde{\varkappa}_n(t)|\tilde{\gamma} z + \tilde{c}\bar{z}| + \tilde{\kappa}_n(t)|\tilde{\gamma} w + \tilde{c}\bar{w}|$$

for $t \geq T$, $|z| \geq R_n$, $|w| \geq R_n$, where

$$\tilde{\varkappa}_n(t) = \frac{|\tilde{\gamma}| + |\tilde{c}|}{|\tilde{\gamma}| - |\tilde{c}|}\left[\hat{\varkappa}(t) + \frac{G(t)}{R_n}\right], \quad \tilde{\kappa}_n(t) = \frac{|\tilde{\gamma}| + |\tilde{c}|}{|\tilde{\gamma}| - |\tilde{c}|}\hat{\kappa}(t).$$

Using (7) end (9) we obtain

$$\tilde{\lambda}_n(t) = \tilde{\lambda}(t) = \frac{|\tilde{\gamma}| + |\tilde{c}|}{|\tilde{\gamma}| - |\tilde{c}|}\left[\hat{\kappa}(t) + |A(t)| + |B(t)|\right].$$

Further we get $\tilde{\Theta}(t) = \operatorname{Re} a_0 - \tilde{\varkappa}(t)$ from (11). Suppose the continuity of θ'. Put $\tilde{\beta}(t) = -\tilde{\lambda}(t)(\theta'(t))^{-1}$, $\tilde{\beta}_n(t) = -\tilde{K}_n e^{-\omega_n(t-T)}$, where $\tilde{K}_n = \omega_n K_n$, $\omega_n, K_n \in \mathbb{R}^0_+$. Clearly $\tilde{\beta}'_n(t) = -\omega_n\tilde{\beta}_n(t)$. Define $\tilde{\Lambda}_n(t) := \tilde{\Theta}_n(t) = \operatorname{Re} a_0 - \tilde{\varkappa}_n(t) + \tilde{\beta}_n(t)$.

Using Theorem 1 and Corollary 2, we obtain the following

Example 2. *Let $t_1 \geq T$ and $R_0 \in \mathbb{R}_+$, $\omega_0 \in \mathbb{R}^0_+$ be such that*

$$\hat{\kappa}(t) + |A(t)| + |B(t)| \leq \theta'(t)K_0\omega_0\frac{|\tilde{\gamma}| - |\tilde{c}|}{|\tilde{\gamma}| + |\tilde{c}|}e^{-\omega_0(t-T)}, \tag{37}$$

$$\hat{\varkappa}(t) + \frac{G(t)}{R_0} \leq \frac{|\tilde{\gamma}| - |\tilde{c}|}{|\tilde{\gamma}| + |\tilde{c}|}\left[(1 - K_0)\omega_0 + \operatorname{Re} a_0\right] \tag{38}$$

for $t \geq T$. Suppose

$$\inf_{t\geq t_1}\left[\operatorname{Re} a_0(t-t_1) - \frac{|\tilde{\gamma}|+|\tilde{c}|}{|\tilde{\gamma}|-|\tilde{c}|}\int_{t_1}^{t}\left(\hat{\varkappa}(s)+\frac{G(s)}{R_0}\right)ds\right] = \nu^*,$$

where $\nu^* \in (-\infty, \infty)$. If a solution $z(t)$ of (36) satisfies $\min_{\theta(t_1)\leq s\leq t_1}|z(t)| > R_0$ and

$$\frac{|\tilde{\gamma}|-|\tilde{c}|}{|\tilde{\gamma}|+|\tilde{c}|}|z(t_1)| - K_0\omega_0(t_1-\theta(t_1))e^{-\omega_0(t_1-T)}\max_{\theta(t_1)\leq s\leq t_1}|z(t)| > R_0e^{K_0-\nu_0},$$

then

$$|z(t)| \geq K^* \exp\left[\operatorname{Re} a_0(t-t_1) + \frac{|\tilde{\gamma}|+|\tilde{c}|}{|\tilde{\gamma}|-|\tilde{c}|}\int_{t_1}^{t}\left(\hat{\varkappa}(s)+\frac{G(s)}{R_0}\right)ds\right],$$

where

$$K^* = \left[\frac{|\tilde{\gamma}|-|\tilde{c}|}{|\tilde{\gamma}|+|\tilde{c}|}|z(t_1)| - K_0\omega_0(t_1-\theta(t_1))e^{-\omega_0(t_1-T)}\max_{\theta(t_1)\leq s\leq t_1}|z(s)|e^{-K_0}\right].$$

If moreover

$$\lim_{t\to\infty}\left[\operatorname{Re} a_0(t-t_1) - \frac{|\tilde{\gamma}|+|\tilde{c}|}{|\tilde{\gamma}|-|\tilde{c}|}\int_{t_1}^{t}\left(\hat{\varkappa}(s)+\frac{G(s)}{R_0}\right)ds\right] = \infty,$$

then for any $\varepsilon > 0$ there exists a $t_2 > t_1$ such that $|z(t)| > \varepsilon$ holds for all $t \geq t_2$ for which the solution $z(t)$ is defined.
Notice that (37) implies (8) with $n = 0$ and (38) implies $\tilde{\beta}_0'(t) \geq \tilde{\Lambda}_0(t)\tilde{\beta}_0(t)$. The number ν from Theorem 1 equals $\nu^* - K_0 - \ln(|\tilde{\gamma}|+|\tilde{c}|)$.

Using Theorem 2 and Remark 2 with $\eta_1(t) \equiv 1$, we get

Example 3. *If the function $\hat{\varkappa}(t)$ is continuous on $[T, \infty$ and there is a constant $C > 0$ such that*

$$\frac{|\tilde{\gamma}|+|\tilde{c}|}{|\tilde{\gamma}|-|\tilde{c}|}\left[\hat{\varkappa}(t)+\hat{\kappa}(t)+|A(t)|+|B(t)|\right] + \frac{|\tilde{\gamma}|+|\tilde{c}|}{C}G(t) < \operatorname{Re} a_0 \tag{39}$$

for $t \geq T$, then there exists a $t_2 > T$ and a solution $z_0(t)$ of (36) satisfying

$$|z_0(t)| \leq \frac{C}{|\tilde{\gamma}|-|\tilde{c}|}\exp\int_T^t[\xi(s)\,ds],$$

where

$$\xi(t) = \operatorname{Re} a_0 - \frac{|\tilde{\gamma}|+|\tilde{c}|}{|\tilde{\gamma}|-|\tilde{c}|}\left[\hat{\varkappa}(t)+\hat{\kappa}(t)+|A(t)|+|B(t)|\right] - \frac{|\tilde{\gamma}|+|\tilde{c}|}{C}G(t).$$

Moreover, if $\lim_{t\to\infty}\int_T^t \xi(s)\,ds < \infty$, then the solution $z_0(t)$ is bounded.
Notice that the condition (39) is equivalent to $\eta_1(t)\tilde{\Lambda}(t) > \theta'(t)|\tilde{\beta}(t)| + C^{-1}\varrho(t)$.

Using Theorem 3 together with Remark 2, where $\eta_1(t) \equiv \frac{1}{2}$ and defining $\xi(t)$ by

$$\xi(t) = \frac{1}{2}\left[\operatorname{Re} a_0 - \frac{|\tilde\gamma| - |\tilde c|}{|\tilde\gamma| + |\tilde c|}\hat\varkappa(t)\right] - \frac{|\tilde\gamma| + |\tilde c|}{|\tilde\gamma| - |\tilde c|}\left[\hat\kappa(t) + |A(t)| + |B(t)|\right] - \frac{|\tilde\gamma| + |\tilde c|}{C}G(t),$$

we get the following

Example 4. *Let $\hat\varkappa(t)$ be continuous on $[T,\infty)$ and there is a constant $C > 0$ such that*

$$\int_T^\infty \hat\varkappa(t)\,dt < \infty, \qquad \int_T^\infty G(t)\,dt < \infty \tag{40}$$

and

$$\frac{|\tilde\gamma| + |\tilde c|}{|\tilde\gamma| - |\tilde c|}\left[\frac{\hat\varkappa(t)}{2} + \hat\kappa(t) + |A(t)| + |B(t)|\right] + \frac{|\tilde\gamma| + |\tilde c|}{C}G(t) < \frac{1}{2}\operatorname{Re} a_0 \tag{41}$$

for $t \geq T$. Let $R_n \in \mathbb{R}_+$, $K_n, \omega_n \in \mathbb{R}_+^0$ and $\tau_n \geq T$ such that $\lim_{n\to\infty} R_n = 0$, $\omega_n > \frac{1}{2}\operatorname{Re} a_0$ for $n \in \mathbb{N}$

$$\limsup_{n\to\infty}\left[K_n e^{-\omega_n(\tau_n - T)} + \frac{1}{R_n}\int_{\tau_n}^\infty G(\sigma)\,d\sigma\right] < \infty, \tag{42}$$

and

$$\hat\varkappa(t) + \frac{G(t)}{R_n} \leq \frac{|\tilde\gamma| - |\tilde c|}{|\tilde\gamma| + |\tilde c|}\left[(1 - K_n)\omega_n + \operatorname{Re} a_0\right], \tag{43}$$

$$\hat\kappa(t) + |A(t)| + |B(t)| \leq \theta'(t)K_n\omega_n\frac{|\tilde\gamma| - |\tilde c|}{|\tilde\gamma| + |\tilde c|}e^{-\omega_n(t-T)} \tag{44}$$

for $t \geq \tau_n$ and $n \in \mathbb{N}$. Then there exists a solution $z_0(t)$ of (36) such that

$$\lim_{t\to\infty}\min_{\theta(t)\leq s\leq t}|z_0(t)| = 0.$$

Notice that (43) and (44) imply (8) and $\tilde\beta_n'(t) \geq \tilde\Lambda_n(t)\tilde\beta_n(t)$, respectively. The condition (41) implies $\eta_1(t)\tilde\Lambda(t) > \theta'(t)|\tilde\beta(t)| + C^{-1}\varrho(t)$, the conditions (40) together with $\omega_n > \frac{1}{2}\operatorname{Re} a_0$ imply (28), (29) and (30).

4 Conclusion

In the present paper we have improved the results presented in [10] under the condition $\liminf_{t\to\infty}\big(|\operatorname{Im} a(t)| - |b(t)|\big) > 0$ instead of $\liminf_{t\to\infty}\big(|a(t)| - |b(t)|\big) > 0$ considered in [10]. We have obtained several results which are similar to the propositions in the related article but they can be more efficient, which is illustrated by Example 1. The applicability of results is illustrated by Examples 2-4. The results dealing with the existence of a bounded solution or solution tending to zero (Examples 3,4) seem to have no analogy for corresponding real systems of two delayed differential equations.

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